A Backward dual representation for the quantile hedging of Bermudan options

Géraldine Bouveret

Imperial College London

joint work with
B. Bouchard (Paris Dauphine) and J.F. Chassagneux (Imperial College London)

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Problem formulation

- $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $\mathbb{F} := \{\mathcal{F}_t, 0 \leq t \leq T\}$ and W a d-dim. BM,

- $\forall\,(t,x) \in [0,T] \times (0,\infty)^d$, $T > 0$ and for $s \geq t$:

$$X_s^{t,x} = x + \int_t^s \mu(r, X_r^{t,x})\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x})\mathrm{d}W_r\,,$$

with

$$\mu : [0,T] \times (0,\infty)^d \to \mathbb{R}^d$$
$$\text{and}\sigma : [0,T] \times (0,\infty)^d \to \mathbb{M}^d \text{ Lipschitz continuous}\,,$$

- σ is invertible and $\lambda := \sigma^{-1}\mu$ is bounded,
- $\mathbb{Q}_{t,x} \sim \mathbb{P}$ is unique and is s.t. $\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}_{t,x}} = Q_{t,x,1}$ where for $s \geq t$:

$$\mathrm{d}Q_{t,x,1}(s) = \lambda(s, X_{t,x}(s))Q_{t,x,1}(s)\mathrm{d}W_s^{\mathbb{Q}_{t,x}} \in (0,\infty),$$
$$Q_{t,x,1}(t) = 1\,.$$

Problem formulation (cont.)

- An admissible financial strategy is a d-dimensional predictable process ν s.t.

$$\mathbb{E}^{\mathbb{Q}_{t,x}}\left[\int_t^T |\nu_r^\top \sigma(r, X_r^{t,x})|^2 dr\right] < \infty\,,$$

and the corresponding wealth process

$$Y^{t,x,y,\nu} := y + \int_t^{\cdot} \nu_r^\top \mathrm{d}X_r^{t,x} \geq 0\,, \quad \text{on } [t, T]\,,$$

given (t, x) and $y \geq 0$.

- $\mathcal{U}_{t,x,y}$ is the collection of admissible financial strategies.

Problem formulation (cont.)

Fix a finite collection of times

$$\mathbb{T}_t := \{t_0 = 0 \leq \cdots \leq t_i \leq \cdots \leq t_n = T\} \cap (t, T],$$

together with non-negative payoff functions

$$x \in (0, \infty)^d \mapsto g(t_i, x), \text{ Lipschitz continuous for all } i \leq n.$$

The quantile hedging problem is

$$v(t, x, p) := \inf \Gamma(t, x, p),$$

where

$$\Gamma(t, x, p) := \left\{ y \geq 0 \ : \ \exists \nu \in \mathcal{U}_{t,x,y} \text{ s.t. } \mathbb{P}\left[\bigcap_{s \in \mathbb{T}_t} \{ Y_s^{t,x,y,\nu} \geq g(s, X_s^{t,x}) \} \right] \geq p \right\}.$$

Problem formulation (cont.)

Remark (Preliminary remarks)

- Meaning of $v(t,\cdot,1)$...

$$v(t,x,1) \;=\; \mathbb{E}^{\mathbb{Q}_{t,x}}[(v \vee g)(t_{i+1}, X^{t,x}_{t_{i+1}}, 1)]\,, \text{ for } t \in [t_i, t_{i+1})\,,$$

with $i < n$ and

$$g(t,x,p) := g(t,x)\mathbb{1}_{\{0<p\leq 1\}} + \infty \mathbb{1}_{\{p>1\}}\,, \quad \text{for } p \in \mathbb{R}\,.$$

- $p \mapsto v(\cdot,p)$ is non-decreasing.
- $v(\cdot,p) = 0$ if $p \leq p_{\min}(t,x)$ where

$$p_{\min}(t,x) := \mathbb{P}[g(s, X^{t,x}_s)\mathbb{1}_{\{s<T\}} = 0 \;\text{ for all } s \in \mathbb{T}_t]\,.$$

Hyp: $p_{\min}(t,\cdot) < 1$, for $t < T \Rightarrow v(t,x,1) > 0\,,$ for $t < T\,.$

What can we find in the literature?

(A) Markovian Framework:

(1) Incomplete market case:

(a) **European Case:** Soner and Touzi in [7] and [8], Bouchard, Elie and Touzi in [2],

(b) **American Case:** Bouchard and Vu in [3],

(2) Complete market case :

(a) **European Case:** Bouchard, Elie and Touzi in [2] and Föllmer and Leukert in [5].

(B) Non-Markovian Framework:

Bouchard, Elie and Reveillac in [1] and Jiao, Klopfenstein and Tankov in [6].

Problem reduction

Before all reduce the initial problem to a standard stochastic target one (see [2])......

To this aim introduce the set $\mathcal{A}_{t,p}$ of square integrable predictable processes such that

$$P^{t,p,\alpha} := p + \int_t^{\cdot} \alpha_r^\top \mathrm{d}W_r \in [0,1]\,, \quad \text{on} \quad [t,T]\,.$$

We denote $\hat{\mathcal{U}}_{t,x,y,p} := \mathcal{U}_{t,x,y} \times \mathcal{A}_{t,p}$.

Proposition

Fix $(t,x,p) \in [0,T] \times (0,\infty)^d \times [0,1]$, *then*

$$\Gamma(t,x,p) = \left\{ \begin{array}{c} y \geq 0 : \exists\,(\nu,\alpha) \in \hat{\mathcal{U}}_{t,x,y,p} \;\; s.t. \\ Y^{t,x,y,\nu} \geq g(\cdot, X^{t,x}) \mathbb{1}_{\{P^{t,p,\alpha}>0\}} \;\; on \;\mathbb{T}_t \end{array} \right\}. \qquad (1)$$

Problem reduction (cont.)

Proof. Obvious at T. Fix $t < T$.

Let $y \in \bar{\Gamma}(t, x, p)$ with $\bar{\Gamma}$ the RHS in (1) and fix $(\nu, \alpha) \in \hat{\mathcal{U}}_{t,x,y,p}$ s.t. $Y^{t,x,y,\nu} \geq g(\cdot, X^{t,x}, P^{t,p,\alpha})$ on $\mathbb{T}_t$.

Then, $\{Y^{t,x,y,\nu} \geq g(\cdot, X^{t,x})\} \supset \{P^{t,p,\alpha} > 0\}$ on $\mathbb{T}_t$. Since $P^{t,p,\alpha} \in [0,1]$ and $\mathbb{1}_{\{P^{t,p,\alpha} > 0\}} \geq P^{t,p,\alpha}$, we have

$$\mathbb{P}\left[\bigcap_{s \in \mathbb{T}_t} \{Y_s^{t,x,y,\nu} \geq g(s, X_s^{t,x})\}\right] \geq \mathbb{P}\left[\bigcap_{s \in \mathbb{T}_t} \{P_s^{t,p,\alpha} > 0\}\right]$$

$$\geq \mathbb{E}\left[P_T^{t,p,\alpha} \prod_{s \in \mathbb{T}_t \setminus \{T\}} \mathbb{1}_{\{P_s^{t,p,\alpha} > 0\}}\right].$$

Noticing that the process $P^{t,p,\alpha}$ is a martingale, for $s \in \mathbb{T}_t$, $\{P_s^{t,p,\alpha} = 0\} \subset \{P_T^{t,p,\alpha} = 0\}$ we obtain $y \in \Gamma(t, x, p)$.

Problem reduction (cont.)

Proof. (cont.) Fix $y \in \Gamma(t,x,p)$ and choose $\nu \in \mathcal{U}_{t,x,p}$ s.t. $p' := \mathbb{P}\left[\bigcap_{s\in\mathbb{T}_t}\{Y_s^{t,x,y,\nu} \geq g(s,X_s^{t,x})\}\right] \geq p$. By the martingale representation theorem, we can find $\alpha \in \mathcal{A}_{t,p'}$ such that

$$\mathbb{1}_{\bigcap_{s\in\mathbb{T}_t}\{Y_s^{t,x,y,\nu}\geq g(s,X_s^{t,x})\}} = P_T^{t,p',\alpha} \geq P_T^{t,p,\alpha} .$$

Modifying appropriately α we have $\alpha \in \mathcal{A}_{t,p}$. Moreover

$$\mathbb{1}_{\{Y_s^{t,x,y,\nu}\geq g(s,X_s^{t,x})\}} \geq P_T^{t,p,\alpha} , \; s \in \mathbb{T}_t .$$

Now take the conditional expectation and use the fact that $P^{t,p,\alpha}$ is a martingale to get

$$\mathbb{1}_{\{Y^{t,x,y,\nu}\geq g(\cdot,X^{t,x})\}} \geq P^{t,p,\alpha} \Leftrightarrow Y^{t,x,y,\nu} \geq g(\cdot,X^{t,x})\mathbb{1}_{\{P^{t,p,\alpha}>0\}} \text{ on } \mathbb{T}_t .$$

Hence, $y \in \bar{\Gamma}(t,x,p)$.

Dynamic programming

A first way to compute the value function v...

Theorem (Dynamic Programming)

Fix $0 \leq i \leq n-1$ *and* $(t,x,p) \in [t_i, t_{i+1}) \times (0,\infty)^d \times [0,1]$,

$$v(t,x,p) = \inf_{\alpha \in \mathcal{A}_{t,p}} \mathbb{E}^{\mathbb{Q}_{t,x}} \left[(v \vee g) \left(t_{i+1}, X^{t,x}_{t_{i+1}}, P^{t,p,\alpha}_{t_{i+1}} \right) \right] .$$

Standard arguments should lead to a characterization of v as a viscosity solution on each interval $[t_i, t_{i+1})$, $i < n$ of

$$\sup_{\alpha \in \mathbb{R}^d} \left\{ \begin{array}{c} -\partial_t \varphi + \alpha^\top \lambda D_p \varphi \\ -\frac{1}{2} \left(\mathrm{Tr}[\sigma\sigma^\top D^2_{xx}\varphi] + 2\,\mathrm{Tr}[\alpha^\top \sigma^\top D^2_{xp}\varphi] + |\alpha|^2 D^2_{pp}\varphi \right) \end{array} \right\} = 0 \,,$$

with the boundary condition

$$v(t_{i+1}-, \cdot) = (v \vee g)(t_{i+1}, \cdot) \,.$$

Dual backward algorithm: intuition of the main result

As with Bouchard, Elie and Touzi in [2] for $n = 1$, take the Fenchel transform $v^\sharp$ of v, i.e.

$$v^\sharp(t, x, q) := \sup_{p \in \mathbb{R}} \left(pq - v(t, x, p)\right) ,$$

to deduce that $v^\sharp$ should be a viscosity solution of the linear PDE on each interval $[t_i, t_{i+1})$, $i < n$ of

$$-\partial_t \varphi - \frac{1}{2}\left(\mathrm{Tr}[\sigma\sigma^\top D^2_{xx}\varphi] + 2q\,\mathrm{Tr}[\lambda^\top \sigma^\top D^2_{xq}\varphi] + |\lambda|^2 q^2 D^2_{qq}\varphi\right) = 0\,,$$

with the boundary condition

$$v^\sharp(t_{i+1}-, \cdot) = (v \vee g)^\sharp(t_{i+1}, \cdot)\,.$$

Dual backward algorithm: intuition of the main result (cont.) and main result

By the Feynman-Kac representation this corresponds to the following backward algorithm for $i < n$

$$\begin{cases} w(T,x,q) & := & q + \infty \mathbb{1}_{\{q<0\}} \,, \\ w(t,x,q) & := & \mathbb{E}^{\mathbb{Q}_{t,x}}\left[(w^\sharp \vee g)^\sharp(t_{i+1}, X^{t,x}_{t_{i+1}}, Q^{t,x,q}_{t_{i+1}})\right] \; t \in [t_i, t_{i+1}) \,, \end{cases}$$

Main result...

Theorem (Main result)

$v = w^\sharp$ *on* $[0,T] \times (0,\infty)^d \times [0,1]$.

Aim: Prove the result by relying on dual arguments only!

The backward algorithm as a lower and upper bound

(1) The backward algorithm as a lower bound

Proposition

$v \geq w^\sharp$ *on* $[0, T] \times (0, \infty)^d \times [0, 1]$.

Proof. Use the definition of the Legendre Fenchel transform and argue by induction.

(2) The backward algorithm as a upper bound

This is the more involved part...

Proposition

$v \leq w^\sharp$ *on* $[0, T] \times (0, \infty)^d \times [0, 1]$.

The backward algorithm as a upper bound

Idea of the proof.
Fix $0 \leq i \leq n-1$ and let $(t, x, p) \in [t_i, t_{i+1}) \times (0, \infty)^d \times [0, 1]$.

Step 1. Prove by induction that a convexification in the dynamic programming algorithm holds, i.e.

$$v(t, x, p) = \inf_{\alpha \in \mathcal{A}_{t,p}} \mathbb{E}^{\mathbb{Q}_{t,x}} \left[\mathrm{co}[v \vee g] \left(t_{i+1}, X^{t,x}_{t_{i+1}}, P^{t,p,\alpha}_{t_{i+1}} \right) \right] .$$

where for a given function f, $\mathrm{co}[f]$ is its closed convex envelope.

Step 2. Prove by induction the probabilistic representation of the dual function, i.e. there exists $\bar{\alpha} \in \mathcal{A}_{t,p}$ such that

$$w^\sharp(t, x, p) = \mathbb{E}^{\mathbb{Q}_{t,x}} \left[\mathrm{co}[w^\sharp \vee g] \left(t_{i+1}, X^{t,x}_{t_{i+1}}, P^{t,p,\bar{\alpha}}_{t_{i+1}} \right) \right] .$$

The backward algorithm as a upper bound (cont.)

To prove **Step 2.** we have, by proceeding backward, to:

(a) prove a decomposition in simple terms of $(w^\sharp \vee g)^\sharp$ and $(w^\sharp \vee g)^{\sharp\sharp}$,

(b) study the subdifferential of $(w^\sharp \vee g)^\sharp$.

(c) find a particular value p in the subdifferential of $w(t_i, \cdot)$,

(d) apply a martingale representation argument between the elements of the subdifferential of $(w^\sharp \vee g)^\sharp$ at t_{i+1} and p at t_i (cf. European case). *Be careful we have to study the limits of $w^\sharp$ in p!*

The backward algorithm as a upper bound (cont.)

Example: fix $(t, x, p) \in [t_{n-1}, T) \times (0, \infty)^d \times [0, 1]$.

(1) Decomposition

We know from (a)

$$w(t, x, q) = \mathbb{E}^{\mathbb{Q}_{t,x}}[(qQ_T^{t,x,1} - g(T, X_T^{t,x}))^+] \\ = \mathbb{E}^{\mathbb{Q}_{t,x}}[g^\sharp\left(t_{i+1}, X_{t_{i+1}}^{t,x}, qQ_{t_{i+1}}^{t,x,1}\right)] .$$

(2) Study of the subdifferential

It can be proved using the Lebesgue theorem that

$$D_q^+ w(t, x, q) = \mathbb{P}[qQ_T^{t,x,1} \geq g(T, X_T^{t,x})] \\ D_q^- w(t, x, q) = \mathbb{P}[qQ_T^{t,x,1} > g(T, X_T^{t,x})] .$$

leading to $D_q^+ w(t, x, \cdot) \geq 0$ if $q \geq 0$ and $D_q^- w(t, x, \cdot) \geq 0$ if $q > 0$, $\lim_{q\uparrow\infty} D_q^+ w(t, x, q) = 1$, $D_q^+ w(t, x, 0) = p_{\min}(t, x)$.

The backward algorithm as a upper bound (cont.)

(3.a) Martingale representation for $p \in (p_{\min}(t,x), 1)$.
From (2) $\exists\, \tilde{q} \in (0,\infty)$ s.t. $(p_{\min}(t,x), 1)$ lies in the subdifferential of $w(\cdot, \tilde{q})$.

This implies that

$$p = \lambda \mathbb{P}[\tilde{q} Q_T^{t,x,1} \geq g(T, X_T^{t,x})] + (1-\lambda)\mathbb{P}[\tilde{q} Q_T^{t,x,1} > g(T, X_T^{t,x})] .$$

with $\lambda \in [0,1]$, lies in the subdifferential of $w(t,x,\cdot)$ at $\tilde{q}$. By the martingale representation theorem, $\exists\, \bar{\alpha} \in \mathcal{A}_{t,p}$ s.t.

$$\lambda \mathbb{1}_{\tilde{q} Q_T^{t,x,1} \geq g(T, X_T^{t,x})} + (1-\lambda)\mathbb{1}_{\tilde{q} Q_T^{t,x,1} > g(T, X_T^{t,x})}$$
$$= p + \int_t^{t_{i+1}} \bar{\alpha}_s^\top \mathrm{d}W_s =: P_{t_{i+1}}^{t,p,\bar{\alpha}} .$$

The backward algorithm as a upper bound (cont.)

Applying [4, Chapter I, Proposition 5.1] we have,

$$\begin{aligned} w^\sharp(t,x,p) &= \tilde{q}p - w(t,x,\tilde{q}) \\ &= \mathbb{E}^{\mathbb{Q}_{t,x}}\left[P^{t,p,\bar{\alpha}}_{t_{i+1}}\tilde{q}Q^{t,x,1}_{t_{i+1}} - g^\sharp\left(t_{i+1}, X^{t,x}_{t_{i+1}}, \tilde{q}Q^{t,x,1}_{t_{i+1}}\right)\right] \\ &= \mathrm{co}[g]\left(t_{i+1}, X^{t,x}_{t_{i+1}}, P^{t,p,\bar{\alpha}}_{t_{i+1}}\right) . \end{aligned}$$

(3.b) Martingale representation for $p \in [0, p_{\min}(t,x)]$ and $p = \{1\}$.
As $[0, p_{\min}(t,x)]$ belongs to the subdifferential of $w(t,x,\cdot)$ at 0 and $p_{\min}(t,x) = D^+_q w(t,x,0)$ we can find $\lambda \in [0,1]$ such that $p = \lambda D^+_q w(t,x,0)$. We then proceed as in (3.a).
The case $p = 1$ requires the study of the limit at $p = 1$ of $w^\sharp$.

For all the technical details feel free to visit our paper on arXiv: http://arxiv.org/abs/1409.8219

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Thank you!